# The Banach-Tarski Paradox

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## 1 Introduction

What is a set? In the late 19th century, Georg Cantor was the first to formally investigate this question, thus founding the study of set theory as a mathematical discipline. Cantor's definition of a set consisted of two intuitive principles. The first, Extensionality, asserts that two sets are equal exactly when they have the same members; the second, Comprehension, asserts that from any condition on objects, one can build a set whose members are exactly those that satisfy it. This elegant formalization of set theory enabled Cantor to prove the first theorems about infinite sets. What could be wrong with such a beautiful theory?

However, shortly afterward, a problem was discovered. Russell considered the property $x \notin x$, and derived from the Comprehension Principle that there should be a set $R$ such that $x \in R \iff x \notin x$; but then $R \in R \iff R \notin R$, which is a contradiction. This famous contradiction, Russell's Paradox, ended all hope for Cantor's simple, intuitive formalization of set theory.

To replace Cantor's problematic principles, Zermelo formulated a list of seven more conservative axioms that would serve as a rigorous base for set theory. They are the axioms of Extensionality, Pairs, Separation, Power Set, Union, Choice, and Infinity. It was soon discovered that two other axioms were needed, and so the axioms of Replacement and Foundation were added. While each of these axioms are statements that seem intuitively clear, the list lacks the elegant symmetry of Cantor's principles. Their best justification is that these were the axioms needed to prove the same theorems that mathematicians wanted to prove, without the sweeping generality that led to paradoxes such as Russell's.

While Zermelo's axioms are sufficient to build a rich theory of sets, with which the rest of mathematics can be formally modeled, it is deeply unsatisfying that we have no reason to think that the axiom system has captured all the properties of sets. We know now that the situation is much worse: Gödel showed that *no* axiom system can completely and consistently decide the truth of all propositions about sets. So we must face the reality that our system can only approximate a hypothetical "complete definition of sets," if one even exists. We can make our system more powerful by adding more complicated axioms, but we quickly reach the limit of our ability to decide which axioms actually agree with our intuition.

Since there is no formal procedure for deciding whether axioms are valid, we have to make decisions based on intuitive judgment of the theorems that can be derived from them. This is not always easy, but we should at least be able to ask that the axioms don't derive theorems that we believe intuitively to be false. In some cases this may involve questioning the axioms, but in others, it may involve questioning our intuition.

The Banach-Tarski Paradox serves to drive home this point. It is not a paradox in the same sense as Russell's Paradox, which was a formal contradiction—a proof of an absolute falsehood. Instead, it is a highly unintuitive theorem: briefly, it states that one can cut a solid ball into a small finite number of pieces, and reassemble those pieces into two balls, each identical in size to the original. As a simple extension, it can be shown that one can start with any bounded set with nonempty interior and reassemble it into any other such set of any volume, so that one could, in principle, begin with a pea and end up with a ball as large as the Sun.

Its proof relies crucially on the Axiom of Choice:

**Axiom of Choice.** ***(AC)****. For any sets $A, B$ and binary relation $P \subseteq A \times B$,*

$$(\forall x \in A)(\exists y \in B)P(x, y) \implies (\exists f \colon A \to B)(\forall x \in A)P(x, f(x)).$$

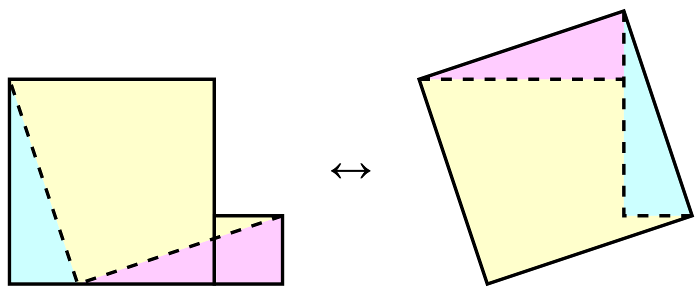
In words, this axiom asserts that if there are arbitrarily many decisions to be made, each of which has at least one possible choice, then there is a function that assigns a choice for each decision. Unlike the other axioms, it is fundamentally nonconstructive in that it does not say anything about *which* choices are made. It is needed for the proof of many important results, such as the comparability of cardinal numbers, or Tychonoff's Theorem that the product of any collection of compact topological spaces is compact.

But the fact that the Axiom of Choice leads to such an unintuitive result as the Banach-Tarski Paradox initially caused many mathematicians to question the inclusion of Choice in our standard list of axioms, just as Russell's paradox had called Cantor's Comprehension Principle into question. Weaker forms of Choice have been proposed to exclude the Banach-Tarski Paradox and similar unintuitive results. A more standard view today is to accept the full generality of Choice; however, it is still common practice to carefully label theorems with **(AC)** when their proof requires the use of Choice.

We begin in the next section by studying the clean theory of polygonal dissections, in order to contrast it with what happens when we allow dissection into arbitrary sets. We give a brief introduction to Lebesgue measure. Then, after a quick detour into some necessary group theory, we construct the Paradox itself. Finally, we conclude with some discussion of the Paradox's implications for measure theory, and for the Axiom of Choice.

## 2 Polygonal Dissections

Dissection and reassembly of a figure is an important concept in classical geometry. It was used extensively by the Greeks to derive theorems about area, including the well-known Pythagorean Theorem (Figure 1).



$$a^2 + b^2 \quad = \quad c^2$$

Figure 1: Pythagorean Theorem

**Definition 1.** *We say that two polygons*[1] *are **congruent by dissection** if they can be dissected (ignoring boundaries) into the same finite number of polygonal pieces, which can be put into correspondence such that each pair of corresponding pieces are congruent (that is, some isometry of the plane sends one to the other).*

The basis for the proofs of the Greeks was that if two polygons are congruent by dissection, then they have the same area. They took this as a defining property of area, in some sense. The converse is also true, as was shown in the early 19th century.

**Theorem 1.** *Any two polygons with the same area are congruent by dissection.*

*Proof.* First, observe that congruence by dissection is an equivalence relation. To see transitivity, suppose that a polygon $Q$ can be dissected along some set of cuts into pieces that can be rearranged to form $P$, and can also be dissected along another set of cuts into pieces that form $R$. Then by making both sets of cuts, we end up with a set of pieces that can be rearranged to form either $P$ or $R$; composing these two procedures, we get that $P$ and $R$ are congruent by dissection.

Now it is sufficient to show that every polygon is congruent by dissection to a square. To do this, dissect the polygon into triangles. Each of these triangles is easily dissected into a rectangle, and then into a square (Figure 2).

To finish, we need to assemble the resulting set of squares into one large square. The construction for this step is exactly the proof of the Pythagorean Theorem, in Figure 1 above.

[1] When we say "polygon," we intend to allow figures composed of disjoint or non-simply-connected parts.

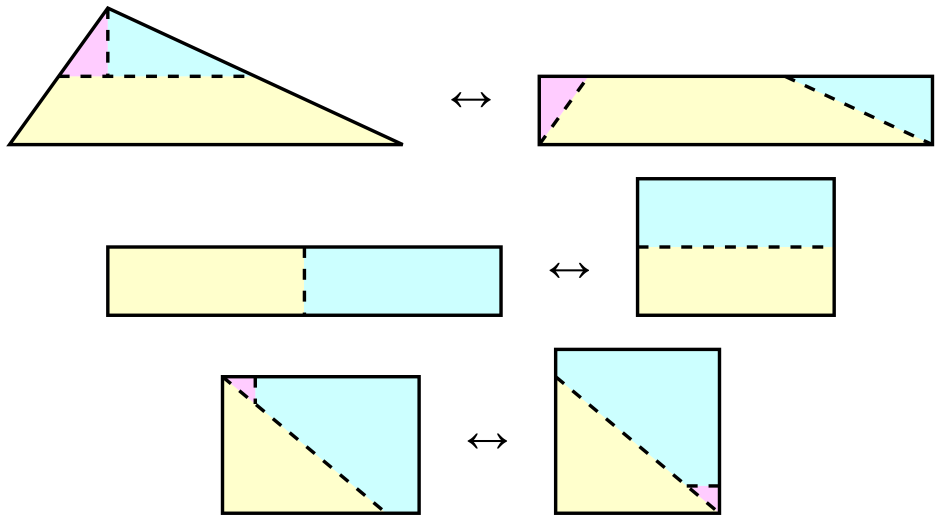


Figure 2: Dissection of a triangle into a square

# 3 Equidecomposability

It may seem somewhat unsatisfying that boundaries are ignored in this discussion. However, since a line segment has zero area, it makes sense that they should not matter. In fact, it can be shown that if we are more careful about the boundaries, it is still possible to perform all of these dissections, using a few extra oddly-shaped (non-polygonal) pieces to absorb unwanted segments. This idea will become clear later.

The analogue of dissection where one does not throw away boundaries is called equidecomposition, and can be defined on arbitrary subsets of $\mathbb{R}^n$, not just polygons.

**Definition 2.** *Two subsets $A, B \subseteq \mathbb{R}^n$ are called **equidecomposable** if they can be partitioned into the same finite number of pieces, which can be matched such that each pair of corresponding pieces are congruent.*

# 4 Lebesgue Measure

In order to formalize our notions of length, area, and volume, we want a definition of **measure** that assigns nonnegative reals (or $\infty$) to subsets of $\mathbb{R}^n$, with the following properties.

1. An $n$-dimensional hypercube of side $x$ has measure $x^n$.

2. The measure of the union of finitely many disjoint figures $A_i \subseteq \mathbb{R}^n$ is the sum of their measures. (One could also ask for countable additivity instead of just

finite additivity, but we will not consider such cases here.)

One such definition is the Lebesgue measure (which is in fact countably additive). However, there is a serious caveat: it will turn out that it is not defined for all sets. The construction of the Lebesgue measure is somewhat technical; one definition is as follows.

**Definition 3.** *For any subset $B \subseteq \mathbb{R}^2$, define $\lambda^*(B)$ to be the infimum of all $k \in [0, \infty]$ such that $B$ can covered by a countable set of hypercubes with total measure $k$. Then we say that $A \subseteq \mathbb{R}^2$ is **Lebesgue measurable** if*

$$\lambda^*(B) = \lambda^*(A \cap B) + \lambda^*(B \setminus A)$$

*for all $B \subseteq \mathbb{R}^n$; if so, its **Lebesgue measure** is defined to be $\lambda(A) = \lambda^*(A)$.*

It is true that all polygons are Lebesgue measurable, as are all circles and a wide variety of other figures, and that line segments have measure 0 in the plane. So we can use Lebesgue measure to formalize all of the proofs about area from classical geometry. However, not all sets are Lebesgue measurable—as we shall see, the Banach-Tarski Paradox is a striking demonstration of this shortcoming of Lebesgue measure, or indeed any measure that purports to satisfy the properties above.

# 5 A Bit of Group Theory

In order to construct the Banach-Tarski Paradox in $\mathbb{R}^3$, we will first need to take a detour into the abstract world of group theory. We generalize the notion of equidecomposability to the action of a group $G$ on a set $X$. (We say that $G$ acts on $X$ if each element $g \in G$ corresponds to an action $X \to X$, written $x \mapsto gx$, such that these actions respect the group laws: $(g_1 g_2)x = g_1(g_2 x)$, $ex = x$. For example, the group of isometries of $\mathbb{R}^n$ acts on $\mathbb{R}^n$.)

**Definition 4.** *If $G$ acts on $X$, we say that $A, B \subseteq X$ are **G-equidecomposable** if they can be partitioned into the same finite number of pieces, which can be matched such that each pair of corresponding pieces $A_i, B_i$ are related by the action of some $g_i \in G$: $B_i = g_i A_i = \{g_i a \mid a \in A_i\}$.*

We recover our original notion of equidecomposability in $\mathbb{R}^n$ when $G$ is the isometry group of $\mathbb{R}^n$. But first we will investigate analogues of the Banach-Tarski in other groups, with the goal of finding one that we can transfer into $\mathbb{R}^3$. An abstract version of the paradox would be a situation where some $A \subset X$ is equidecomposable with $B = g_1 A \cup g_2 A$, where $g_1 A$ and $g_2 A$ are two disjoint copies of $A$.

As a trivial example, we could let $X = \mathbb{Z}$ and $G$ be the group of all bijections $\mathbb{Z} \to \mathbb{Z}$. Then we can see that $A = \mathbb{N}$ and $B = \mathbb{Z}$ are $G$-equidecomposable, by letting $A = E \cup O = \{0, 2, 4, \dots\} \cup \{1, 3, 5, \dots\}$ and $B = \mathbb{N} \cup \mathbb{Z}^-$, with bijections $E \to \mathbb{N}$

defined by $e \mapsto \frac{e}{2}$, and $O \mapsto \mathbb{Z}^-$ defined by $o \mapsto -\frac{o+1}{2}$. So the integers can be put into correspondence with the natural numbers, which is neither surprising nor particularly unsettling.

To get a more interesting "paradox," we consider free groups.

**Definition 5.** *The **free group** on $k$ generators $\sigma_1, \ldots, \sigma_k$ is the group of finite words on the symbols $\sigma_1, \ldots, \sigma_k, {\sigma_1}^{-1}, \ldots, {\sigma_k}^{-1}$ under composition, modulo the equivalences $\sigma_i {\sigma_i}^{-1} = {\sigma_i}^{-1}\sigma_i = \epsilon$ (the empty word).*

In particular, we are going to show that the free group $F$ on two generators is equidecomposable with two copies of itself, under its own group action (any group acts on itself by left multiplication). In order to talk about two disjoint copies of $F$, we expand our focus to the group $G = F \times \{1, r\}$ $(r^2 = 1)$ under its own group action.

**Theorem 2.** *Consider the group $G = F \times \{1, r\}$ $(r^2 = 1)$ acting on itself, where $F$ is the free group on two generators. Then $F \times \{1\}$ is $G$-equidecomposable with all of $G$. (Less formally, one copy of $F$ is equidecomposable with two copies of $F$.)*

The most straightforward approach uses the following generalization of the Schröder-Bernstein Theorem from set theory.

**Theorem 3** (Banach-Schröder-Bernstein Theorem)**.** *Suppose group $G$ acts on $X$. If each of $A, B \subseteq X$ is $G$-equidecomposable with a subset of the other, then $A$ is $G$-equidecomposable with $B$.*

*Proof.* Consider the two given $G$-equidecompositions as maps $\alpha\colon A \to \alpha A \subseteq B$ and $\beta\colon B \to \beta B \subseteq A$. Let $A_1 = \bigcup_{i=0}^{\infty} (\beta\alpha)^i (A \smallsetminus \beta B)$; since $\beta\alpha A \subseteq \beta B \subseteq A$, we have by induction that $(\beta\alpha)^i A \subseteq A$, so $A_1 \subseteq A$. Let $A_2 = A \smallsetminus A_1$, $B_1 = \alpha A_1 \subseteq B$, $B_2 = B \smallsetminus B_1$. Observe that $A_1 = (A \smallsetminus \beta B) \cup \beta\alpha A_1$, so $A_2 = \beta B \cap (A \smallsetminus \beta\alpha A_1) = \beta B \smallsetminus \beta\alpha A_1 = \beta B \smallsetminus \beta B_1 = \beta B_2$. We can therefore construct a $G$-equidecomposition of $A$ and $B$ by cutting $A_1$ from $A_2$, then performing $\alpha$ on $A_1$ and $\beta^{-1}$ on $A_2$.

*Proof of Theorem 2.* Write $F$ as the free group on $\sigma, \tau$. The identity map is a $G$-equidecomposition from $F \times \{1\}$ to a subset of $G$. To construct a $G$-equidecomposition in the other direction, define $W(\rho)$ for any symbol $\rho$ to be the set of words (in shortest form) that begin with $\rho$, and write $G = G_1 \cup G_2 \cup G_3 \cup G_4$, where

$$\begin{aligned}
G_1 &= W(\sigma) \times \{1\} & F_1 &= (\epsilon, 1)G_1 = W(\sigma) \times \{1\}, \\
G_2 &= (F \smallsetminus W(\sigma)) \times \{1\} & F_2 &= (\sigma^{-1}, 1)G_2 = W(\sigma^{-1}) \times \{1\}, \\
G_3 &= W(\tau) \times \{r\} & F_3 &= (\epsilon, r)G_3 = W(\tau) \times \{1\}, \\
G_4 &= (F \smallsetminus W(\tau)) \times \{r\} & F_4 &= (\tau^{-1}, r)G_2 = W(\tau^{-1}) \times \{1\}.
\end{aligned}$$

Then $F_1, F_2, F_3, F_4$ are disjoint subsets of $F \times \{1\}$. By the Banach-Schröder-Bernstein Theorem, $F \times \{1\}$ is $G$-equidecomposable with $G$.

## 6 From Free Groups to $\mathbb{R}^3$

This paradox with the free group of two generators is particularly interesting to us, because it turns out that the free group on two generators can be realized as a group of rotations of the sphere.

**Theorem 4.** *There exist rotations $\phi, \rho$ of the unit sphere in $\mathbb{R}^3$ that are **independent**; i.e., no nontrivial word on $\phi, \rho, \phi^{-1}, \rho^{-1}$ corresponds to the identity rotation. Consequently, these rotations generate the free group $F$ on two generators.*

*Proof.* There are several ways to find such rotations. In fact, it is not hard to prove topologically that almost every pair of rotations will do. One constructive solution is to use rotations of angle $\arccos \frac{1}{3}$ about perpendicular axes:

$$\phi = \begin{bmatrix} \frac{1}{3} & -\frac{2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{bmatrix}, \qquad \rho = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{3} & -\frac{2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{bmatrix}.$$

Then it can be shown that any word $w$ on $\phi, \rho, \phi^{-1}, \rho^{-1}$ sends $(1, 0, 0)$ to a vector of the form $(a, b\sqrt{2}, c)/3^k$, where $k$ is the length of the word $w$ and $b$ is not divisible by 3. In particular, $(1, 0, 0)$ is never sent to itself by a nontrivial word, so the rotations are independent.

With this representation of the free group $F$ in hand, we are ready to begin our construction of the Banach-Tarski Paradox.

## 7 Construction of the Paradox

Our goal is to show that the unit ball $B^3$ is equidecomposable with two unit balls under the isometry group of $\mathbb{R}^3$. We will show how to do this for a spherical shell $S^2$. Then we can view most of the ball as a collection of spherical shells of radii $0 < r < 1$, and apply the same transformations to each shell. This lets us take care of every point except the center. Our first task, then, is to show how to remove the center.

**Lemma 5.** *$B^3$ is equidecomposable with $B^3 \setminus \{0\}$, where $0$ is the ball's center.*

*Proof.* Consider a small circle that passes through 0 and is contained entirely in $B^3$. Let $\rho$ be a 1-radian rotation of this circle. Then the points $0, \rho 0, \rho^2 0, \rho^3 0, \ldots$ are distinct, and if we apply $\rho$ to just this set of points, we get the same set except with 0 missing. This yields a two-piece equidecomposition of $B^3$ with $B^3 \setminus \{0\}$: one piece is $\{0, \rho 0, \rho^2 0, \rho^3 0, \ldots\}$ going to $\{\rho 0, \rho^2 0, \rho^3 0, \ldots\}$ under the rotation $\rho$, and the other piece is the rest of the ball going to itself.

Our next task is a slightly more elaborate version of the same idea. We will need to remove countably many troublesome points from the spherical shell.

**Lemma 6.** *Let $D$ be the set of points on sphere $S^2$ that are fixed by some nontrivial element of $F$ (the free group constructed in Theorem 4). Then $S^2$ is equidecomposable with $S^2 \smallsetminus D$.*

*Proof.* We can use essentially the same trick as in Lemma 5. First note that $D$ is countable, because $F$ is a countable set of words, and each nontrivial rotation of $F$ fixes only the two points of $S^2$ on the rotation's axis. Therefore, there is an axis through the center that does not pass through any point of $D$. Furthermore, there exists a rotation $\rho$ about this axis such that $D, \rho D, \rho^2 D, \rho^3 D, \ldots$ are disjoint, because there are at most countably many rotations $\rho$ about this axis that satisfy $\rho^i d = d'$ for some $i \in \mathbb{Z}^+$, $d, d' \in D$. If we apply this rotation to just the set $D \cup \rho D \cup \rho^2 D \cup \rho^3 D \cup \cdots$, the result is that $D$ disappears. So as in the last proof, we get a two-piece equidecomposition of $S^2$ with $S^2 \smallsetminus D$.

Finally, we use the Axiom of Choice to make the critical step of duplicating $S^2 \smallsetminus D$.

**Lemma 7.** ***(AC).*** *$S^2 \smallsetminus D$ can be partitioned into two sets, each of which is equidecomposable with $S^2 \smallsetminus D$.*

*Proof.* First, note that distinct rotations $f_1, f_2 \in f$ send any point $x \in S^2 \smallsetminus D$ to different images, because if $f_1 x = f_2 x$, then $(f_1 {f_2}^{-1})x = x$, which contradicts $x \notin D$.

The ***F*-orbit** of a point $x \in S^2 \smallsetminus D$ is defined to be the set $Fx = \{fx \mid f \in F\}$ of all such images of $x$. These $F$-orbits partition $S^2 \smallsetminus D$, because they are equivalence classes under the equivalence relation $x \sim y \iff y \in Fx$. By the Axiom of Choice, there is a set $M$ consisting of exactly one member from each $F$-orbit. Then every point $x \in S^2 \smallsetminus D$ can be written uniquely as $x = fm$ for some $f \in F, m \in M$. So the sets $fM$ $(f \in F)$ also partition $S^2 \smallsetminus D$.

By Theorem 2, $F$ is equidecomposable with two copies of itself—that is, $F$ can be partitioned into two subsets $F_1, F_2$ that are each $F$-equidecomposable with $F$. The idea, then, is to look at the actions that these equidecompositions perform on the points $f$, and apply equivalent actions to the sets $fM$.

In particular, let $\phi_i \colon F_i \to F$ be $F$-equidecompositions $(i = 1, 2)$, which we think of as maps $f \mapsto \phi_i f$. The sets $F_1 M, F_2 M$ partition $FM = S^2 \smallsetminus D$ (by the uniqueness of the representation $x = fm$), and we claim that there are equidecompositions $\overline{\phi}_i \colon F_i M \to FM = S^2 \smallsetminus D$ given by $fm \mapsto (\phi_i f)m$.

To see that $\overline{\phi}_i$ is actually an equidecomposition, let $A_{ik} \subseteq F_i$ be a piece that goes to $B_{ik} \subseteq F$ under the $F$-equidecomposition $\phi_i$, by the action of $\phi_{ik} \in F$. Then there is a corresponding piece of $\overline{\phi}_i$: the action of $\phi_{ik}$ sends the piece $A_{ik} M$ to $B_{ik} M$.

Putting everything together, we have the Banach-Tarski Paradox.

**Theorem 8** (Banach-Tarski Paradox)**.** ***(AC).*** *The unit ball $B^3$ is equidecomposable with two copies of itself.*

*Proof.* Combining Lemmas 6, 7, and 6 again, we see that $S^2$ is equidecomposable with two copies of itself. Scaling this construction about the center allows us to do this for the spherical shells of radius $r$, for any $0 < r < 1$, with the same rotations. If we do this for all these shells simultaneously, we get an equidecomposition of $B^3 \setminus \{0\}$ with two copies of itself. Finally, using Lemma 5 before and after this, we see that $B^3$ is equidecomposable with two copies of itself. ☺

With some care, it can be shown that this equidecomposition can be carried out using just five pieces. Somehow, making some cuts with the Axiom of Choice has allowed us to completely subvert the intuitive properties of volume.

As promised, it is now a simple extension to equidecompose any two bounded sets with nonempty interior.

**Corollary 9** (Banach-Tarski Paradox, Strong Form). ***(AC).*** *If $A, B \subset \mathbb{R}^3$ are any bounded sets with nonempty interior, then $A$ and $B$ are equidecomposable.*

*Proof.* We will apply the Banach-Schröder-Bernstein Theorem; to do this, it is sufficient to prove that $A$ is equidecomposable with a subset of $B$ (and vice versa). Let $L$ be an open ball in $B$. By Theorem 8 and induction, $L$ is equidecomposable with any finite number of copies of $L$. Since $A$ is bounded, we can position finitely many copies of $L$ such that they cover $A$. Then $A$ is the disjoint union of subsets of these copies of $L$, hence is equidecomposable with the preimage in $B$ of these subsets under the copying procedure. ☺

# 8 Conclusions

We have derived a completely unintuitive, somewhat disturbing result. What are we to conclude from this?

At first glance, the Banach-Tarski Paradox appears to be an actual contradiction with measure theory, for if we measure all the pieces used in the construction, they must add up both to the volume of the ball and to twice the volume of the ball, and these are different positive real numbers. The resolution of this paradox is that the pieces must not be Lebesgue measurable—or more generally, any function that satisfies the properties of measure listed above cannot be defined on these pieces.

This is perhaps more disturbing when we translate this result from measure theory to the closely related probability theory. Let $A$ be one of the non-measurable pieces of $B^3$. If we pick a random point in $B^3$, clearly either it is in $A$ or it is not. Intuitively, one should be able to assign a probability to the event of the point being in $A$, and this probability should be invariant under rotations of the ball. However, if we tried to add up the probabilities assigned to all the pieces, before and after the rotations, we would get a contradiction. We are forced to admit the existence of an event with undefined probability.

Should we accept the Paradox, or conclude that one of our assumptions is wrong? It is clear that the blame, if we are to find something to blame, rests squarely on the Axiom of Choice. In fact, it has been proved that the other axioms cannot yield a result of this kind. During the controversy over Choice that resulted from the Banach-Tarski Paradox, many proposed to replace Choice with a weaker form. The other axioms are already sufficient to show that the statement of Choice holds when the set $A$ of decisions to be made is finite. A very weak version of Choice is the Countable Principle of Choice, which asserts that the statement holds for countable sets $A$. A slightly stronger version is the Axiom of Dependent Choices.

**Axiom of Dependent Choices.** *For any set $A$ and binary relation $P \in A \times A$,*

$$\begin{aligned} &a \in A \text{ and } (\forall x \in A)(\exists y \in A) P(x, y) \\ &\qquad \implies (\exists f \colon \mathbb{N} \to A)[f(0) = a \text{ and } (\forall n \in \mathbb{N}) P(f(n), f(n+1))]. \end{aligned}$$

All of these propositions are special cases of the Axiom of Choice that are not strong enough to imply the Banach-Tarski Paradox, but good enough for many results (but not all of them: cardinal comparability and the general Tychonoff Theorem require the full form of Choice, and in fact are equivalent to it).

Not all consequences of the Axiom of Choice in measure theory are negative. We have seen a paradox in $\mathbb{R}^3$, and it is clear how to extend it to higher dimensions. However, Banach and Tarski also proved a different result for lower dimensions; they showed that a paradox of this nature cannot exist in $\mathbb{R}^1$ or $\mathbb{R}^2$. Thus they gave a sort of converse to Theorem 1: any two planar polygons that are congruent by dissection (or, equivalently, equidecomposable) must have the same area. This result also requires the Axiom of Choice, arguably in a much more fundamental way.

Ultimately, the Axiom of Choice remains essentially accepted, and the Banach-Tarski Paradox did not result in the same kind of earthshaking changes to the foundations of set theory that Russell's Paradox did. However, the Paradox remains important as a forceful illustration of the sometimes highly counter-intuitive nature of sets, and as such, it has greatly strengthened our understanding of mathematics.

# References

[1] Stan Wagon. *The Banach-Tarski Paradox.* Cambridge University Press, 1985.