Induction and coinduction

Induction:

- construction from the basis
- the least solution $X^\star$ of an inequality: $f(X) \subseteq X$ (closure property)
- proof method: $f(Y) \subseteq Y \implies X^\star \subseteq Y$: $X^\star$ has property Y.

Coinduction:

- destruction from the whole
- the greatest solution $X^\star$ of an inequality: $X \subseteq f(X)$
- proof method: $Y \subseteq f(Y) \implies Y \subseteq X^\star$: Y has property $X^\star$.

Actually, coinduction can be viewed as induction on the complement, let $Y = \overline{X}$, and let $g(Y) = \overline{f(\overline{Y})}$, then $X \subseteq f(X)$ can be reformalized as $g(Y) \subseteq Y$. The least solution of Y is the greatest solution of X for $X \subseteq f(X)$.

Examples

Defining the set of points that are reachable from a point w_0 in a model as the least solution of X such that $f(X) \subseteq X$ where:

$$f(X) = \{w \mid w_0 \to w\} \cup \{w \mid \exists v \in X : v \to w\}$$

Defining the set of points that have infinite descending chains in a model as the greatest solution of X such that $X \subseteq f(X)$ where:

$$f(X) = \{w \mid \exists v \in X : w \to v\}$$

A modal μ-calculus formula: $\nu X.\Diamond X$ (where ν is the greatest fixed point operator) can thus define all the points that have infinite descending chains.

Induction and coinduction

To guarantee those least/greatest solutions do exist:

Lemma

Let $\mu = \bigcap\{X \mid f(X) \subseteq X\}$ and $\nu = \bigcup\{X \mid X \subseteq f(X)\}$. If f is a order-preserving (monotone) function ($X \subseteq Y \implies f(X) \subseteq f(Y)$) then $f(\mu) \subseteq \mu$ and $\nu \subseteq f(\nu)$.

Based on this we can show:

Theorem (Knaster-Tarski, on power sets over W)

If f is a monotone function on subsets of U: $\mathcal{P}(W) \to \mathcal{P}(W)$ then

- *μ is the least fixed point of f.*
- *ν is the greatest fixed point of f.*

Moreover, we can reach μ and ν by (transfinite) iteration of f from $\emptyset$ or W respectively (Kleene's fixed point theorem.)